Quantum dynamics with two Planck constants and the semiclassical limit

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bulletin board ref.: quant-ph/9503023

March 28, 1995

Abstract

The mathematical possibility of coupling two quantum dynamic systems having two different Planck constants, respectively, is investigated. It turns out that such canonical dynamics are always irreversible. Semiclassical dynamics is obtained by letting one of the two Planck constants go to zero. This semiclassical dynamics will preserve positivity, as expected, so an improvement of the earlier proposals by Aleksandrov and by Boucher and Traschen is achieved. Coupling of quantized matter to gravity is illustrated by a simplistic example.

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More then 100 years ago, fundamental theory of the physics world was represented by classical canonical dynamics. Twentieth century brought curious changes: new quantum dynamics was introduced. The apparatus of quantum dynamics is rather unusual. Nevertheless, a universal rule of *quantization* has been invented to construct the quantum counterpart of a given classical canonical dynamics. Classical canonical coordinates $q_1, q_2, \dots$ and momenta $p_1, p_2, \dots$ must be considered hermitian operators, and the classical Poisson bracket

$$\{A,B\}_P \equiv \sum_n \Big(\frac{\partial A}{\partial q_n}\frac{\partial B}{\partial q_n} - \frac{\partial A}{\partial p_n}\frac{\partial B}{\partial p_n}\Big) \tag{1}$$

of dynamic variables $A(q,p),\ B(q,p)$ must be replaced by a quantum bracket:

$$\{A,B\}_Q \equiv -\frac{i}{\hbar}[A,B]. \tag{2}$$

The quantum counterpart of classical Liouville equation of motion $\dot\rho = \{H,\rho\}_P$ will then be the Schrödinger (or von Neumann) equation

$$\dot\rho = \{H,\rho\}_Q \tag{3}$$

where $H(q,p)$ is the Hamiltonian and ρ is the density operator.

Let us outline the opposite way, i.e., obtaining classical dynamics from a given quantum one. Basically, one takes the limit $\hbar \to 0$ and introduces an (overcomplet) basis of normalized states $|q,p\rangle$ for the prospective classical system. These states are wave packets with center q,p. In the limit $\hbar \to 0$, they can and will be chosen (asymptotically) zero-spread in *both* q and p. Taking this basis to work in, all dynamic variables $A(q,p)$ gets asymptotically diagonal so their diagonal elements will be identified as the corresponding classical variables:

$$\lim_{\hbar\to 0}\langle q,p|A|q,p\rangle = A(q,p). \tag{4}$$

In a similar way the density operator will be assumed (asymptotically) diagonal in the basis $|q,p\rangle$; its diagonal $\rho(q,p)$ corresponds to the phase space distribution of the classical system. Expanding the quantum commutator (2) in the leading order in $\hbar$ one can prove the asymptotic equation

$$\lim_{\hbar\to 0}\langle q,p|\{A,B\}_Q|q,p\rangle = \{A(q,p),B(q,p)\}_P. \tag{5}$$

In this sense can one identify classical counterpart of a given quantum dynamics by the limit $\hbar \to 0$ of the latter.

Nowadays, quantum dynamics is thought the fundamental one and classical dynamics is considered a special limit of it. Nevertheless, there is at least one classical dynamics whose quantization is still problematic. As a matter of fact, no experimental evidence up to now has indicated that *gravitation* would be a quantum dynamics. And even theoretical projects of quantizing the classical equations of gravity have failed to be conclusive enough. So, gravitation could happen to be classical. Then, the classical dynamics of gravitation would couple with the quantum dynamics of other fields. This mathematical problem is not at all trivial. For instance, Aleksandrov's semiclassical dynamics [1] was found [2] to violate trivial conditions of positivity. There is no consistent theory to couple classical and quantum systems together [3].

In this Letter, we make an attempt to settle the problem. Our principal aim is to discuss if classical and quantum canonical dynamics could be coupled *at all* in a consistent mathematical scheme. We would not intend to discuss interpretation of the obtained results. Yet, we anticipate the basic lesson. It seems unavoidable that a coupled classical-quantum dynamics be *irreversible* which is in pronounced contrast to the pure classical or pure quantum canonical dynamics though it is less strange from viewpoint of quantum measurement theory [4].

We start with a system composed of two canonical subsystems and we assume *different* Planck constants for each subsystems so that, e.g., $[q_1, p_1] = i\hbar_1$ and $[q_2, p_2] = i\hbar_2$ with, say, $\hbar_1 > \hbar_2$. In the end, we shall take the limit $\hbar_1 \to \hbar$, $\hbar_2 =\to 0$ and in such a way shall we obtain a (hybrid) *semiclassical* dynamics where (q_1, p_1) are quantum and (q_2, p_2) are classical. We prefer this indirect way in order to utilize a powerful classification of quantum equations of motion, due to Lindblad [5, 6]. To our knowledge, no classification is available concerning semiclassical dynamics directly.

Coming back to the two $\hbar$'s, we emphasize that trading with such parametric freedom is mathematically trivial and does not lead to any conflict as long as the states ρ_n of the two systems $(n = 1, 2)$ evolve independently of each other according to their Schrödinger equations [cf. Eq. (3)]: $\dot{\rho}_n = -(i/\hbar_n)[H_n, \rho_n]$ where $H_n = H_n(q_n, p_n)$ are the corresponding Hamiltonians.

Let us turn to the case of interaction. The total Hamiltonian takes the form $H = H_1 + H_2 + H_I$ and the interaction Hamiltonian can in general be

expanded into a series of interacting "currents":

$$H_I(q_1,p_1,q_2,p_2)=\sum_\alpha J_1^\alpha(q_1,p_1)J_2^\alpha(q_2,p_2), \qquad (J_1^\alpha, J_2^\alpha \neq 1). \tag{6}$$

In fact, all dynamic variables of the composed system can be decomposed in a similar form: $A(q_1,p_1,q_2,p_2)=\sum_\alpha A_1^\alpha(q_1,p_1)A_2^\alpha(q_2,p_2)$. Hereafter, until the last part of the Letter, I will spare notations of summation signs and indices so that I write $H_I=J_1J_2$ and $A=A_1A_2$, $B=B_1B_2$ e.t.c..

Now, let us find a suitable generalization of the quantum bracket (2) for the case of the composed system with the two Planck constants. Let us try the ansatz

$$\{A,B\}_Q\equiv-\frac{i}{2\hbar_1}[A_2,B_2]_+[A_1,B_1]-\frac{i}{2\hbar_2}[A_1,B_1]_+[A_2,B_2]. \tag{7}$$

Then the Schrödinger equation of motion (3) could be written in the following form:

$$\dot\rho=-\frac{i}{\hbar_1}[H_1,\rho]-\frac{i}{\hbar_2}[H_2,\rho]-\frac{i}{\hbar_{av}}[H_I,\rho]+\frac{i}{2}\Delta\hbar^{-1}\left(J_1\rho J_2-J_2\rho J_1\right) \tag{8}$$

where $\hbar_{av}=2\hbar_1\hbar_2/(\hbar_1+\hbar_2)$ and $\Delta\hbar^{-1}=\hbar_2^{-1}-\hbar_1^{-1}$.

The first three terms on the RHS of Eq. (8) generate unitary evolution which is however distorted by the fourth term. It would not be a problem if it did not violate positivity of ρ. But it does. Seemingly, we should not ask too sharp questions in the presence of the dynamics $\hbar_1\neq\hbar_2$. We should only inquire about blurred values of the dynamic variables. Consequently, a certain smoothening mechanism must be built in *by hand*. Let us replace the interaction Hamiltonian (6) by a noisy one:

$$H_I^{noise}(q_1,p_1,q_2,p_2)=\Big(J_1(q_1,p_1)+\delta J_1(t)\Big)\Big(J_2(q_2,p_2)+\delta J_2(t)\Big) \tag{9}$$

where $\delta J_1,\delta J_2$ are *classical* noises superposed on the interacting "currents" J_1,J_2. We choose their correlations as follows:

$$\Big\langle\delta J_n(t')\delta J_n(t)\Big\rangle_{noise}=\frac{1}{2}h_{av}^2\Delta\hbar^{-1}\lambda_n\delta(t'-t), \quad (n=1,2), \tag{10}$$

with $\lambda_1\lambda_2=1$ which will be justified later. Assume, for simplicity's, the two noises $\delta J_1,\delta J_2$ are independent of each other. The total Hamiltonian becomes noisy: $H^{noise}=H_1+H_2+H_I^{noise}$. The blurred dynamics is defined by the Schrödinger equation of motion (8) *averaged* over the noise:

$$\dot\rho=\Big\langle\{H^{noise,\rho}\}_Q\Big\rangle_{noise}. \tag{11}$$

Such noisy Hamiltonians are known to generate typical double-commutator terms [6] so the given H^{noise} yields:

$$\dot{\rho} = \{H, \rho\}_Q - \frac{1}{4}\Delta\hbar^{-1}\Big(\lambda_2[J_1,[J_1,\rho]\ ] + \lambda_1[J_2,[J_2,\rho]\ ]\Big). \tag{12}$$

By using the notation $F = \lambda_2^{1/2}J_1 - i\lambda_1^{1/2}J_2$ and by substituting the definition (7) of the generalized quantum bracket into the above equation one obtains:

$$\dot{\rho} = -\frac{i}{\hbar_1}[H_1,\rho] - \frac{i}{\hbar_2}[H_2,\rho] - \frac{i}{\hbar_{av}}[H_I,\rho] - \frac{1}{4}\Delta\hbar^{-1}\left(F^\dagger F\rho + \rho F^\dagger F - 2F\rho F^\dagger\right), \tag{13}$$

provided $\lambda_1\lambda_2 = 1$. This equation belongs to the class of Lindblad master equations [5, 6] so it can be embedded into an enlarged unitary dynamics. This mathematical correspondence assures the consistency of the above master equation though this equation is *not* reversible anymore. Its detailed form is:

$$\begin{aligned}\dot{\rho} = -\frac{i}{\hbar_1}[H_1,\rho] - \frac{i}{\hbar_2}[H_2,\rho] - \frac{i}{\hbar_{av}}[H_I,\rho] \\ -\frac{1}{2}\Delta\hbar^{-1}\Big(iJ_2\rho J_1 - iJ_1\rho J_2 + \frac{1}{2}\lambda_2[J_1,[J_1,\rho]\ ] + \frac{1}{2}\lambda_1[J_2,[J_2,\rho]\ ]\Big).\end{aligned} \tag{14}$$

This master equation will be utilized to construct semiclassical dynamics. According to what we outlined in the first part, to make the system q_2, p_2 classical we need the limit $\hbar_2 \to 0$ and we must work in the asymptotic basis of zero-spread wave packets $|q_2, p_2\rangle$. Also, we assume the density operator ρ is diagonal in this basis:

$$\rho = \int \rho(q_2,p_2)|q_2,p_2\rangle\langle q_2,p_2|dq_2dp_2. \tag{15}$$

Since all dynamic variables are (asymptotically) diagonal in the same basis $|q_2, p_2\rangle$ one expects Eq. (14) to preserve the diagonal form (15) of ρ.

Before taking the limit $\hbar_2 \to 0$ we re-scale the $\lambda-$coefficients: $\lambda_2 = \lambda\hbar_2,\ \ \lambda_1 = \lambda^{-1}/\hbar_2$ otherwise the corresponding terms would diverge. Perform now the limit $\hbar_2 \to 0$ on the diagonal element $\langle q_2, p_2|\dots|q_2, p_2\rangle$ of the quantum master equation (14) and apply the Eq. (5) in it. The resulting semiclassical master equation reads:

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \frac{1}{2}\{H,\rho\}_P - \frac{1}{2}\{\rho,H\}_P - \frac{1}{4}\lambda[J_1,[J_1,\rho]] + \frac{1}{4}\lambda^{-1}\{J_2,\{J_2,\rho\}_P\}_P \tag{16}$$

where, in the total Hamiltonian $H(q_1,p_1,q_2,p_2)$, the variables q_1,p_1 are operators while the variables q_2,p_2 are numbers. Obviously, J_1 is operator and J_2 is number. The object ρ above stands for the diagonal part $\rho(q_2,p_2)$ introduced in Eq. (15). It is density operator of the quantum subsystem *and* phase space distribution of the classical subsystem. Its trace over the quantum subsystem's states yields the phase space distribution $\rho_2(q_2,p_2)$ of the classical subsystem while its integral over q_2,p_2 yields the reduced density operator ρ_1 of the quantum subsystem. One expects that the consistency of the quantum dynamics $\hbar_1\neq\hbar_2$, assured by its Lindblad structure, survives in the semiclassical limit. Hence we claim that Eq. (16) is a consistent one and it preserves the positivity of ρ which was not the case in the earlier proposals [1, 2, 3] to couple classical and quantum canonical dynamics together.

Let us remember the original form (6) of the interaction Hamiltonian and restore the hidden indices and signs of summations in Eq. (16):

$$\begin{aligned}\dot\rho = -\frac{i}{\hbar}[H,\rho]+\frac{1}{2}\{H,\rho\}_P-\frac{1}{2}\{\rho,H\}_P\\ -\frac{1}{4}\sum_{\alpha,\beta}\lambda_{\alpha\beta}[J_1^\alpha,[J_1^\beta,\rho]]+\frac{1}{4}\sum_{\alpha,\beta}\lambda^{-1}_{\alpha\beta}\{J_2^\alpha,\{J_2^\beta,\rho\}_P\}_P\end{aligned} \tag{17}$$

where λ is positive matrix controlling the statistics of noises needed for consistency. Note that this semiclassical master equation might have been derived directly by blurring the Aleksandrov equation [1]:

$$\dot\rho=\left\langle -\frac{i}{\hbar}[H^{noise},\rho]+\frac{1}{2}\{H^{noise},\rho\}_P-\frac{1}{2}\{\rho,H^{noise}\}_P\right\rangle_{noise} \tag{18}$$

with the noisy Hamiltonian (9) of correlations:

$$\left\langle\delta J_1^\alpha(t')\delta J_1^\beta(t)\right\rangle_{noise}=\frac{1}{2}\lambda^{-1}_{\alpha\beta}\delta(t'-t),\quad \left\langle\delta J_2^\alpha(t')\delta J_2^\beta(t)\right\rangle_{noise}=\frac{\hbar^2}{2}\lambda_{\alpha\beta}\delta(t'-t). \tag{19}$$

Finally, let us see an illustrative example. One might consider simple models possessing exact solutions (like, e.g., harmonic coupling between quantum and classical oscillators). Instead of doing so, we recall the original issue as to couple quantized matter with classical gravitation. So we apply the semiclassical master equation (17) to the interaction of quantized nonrelativistic matter with weak classical gravitational field. The quantum subsystem's Hamiltonian is $H_m(q,p)$ with conjugate variables q_n,p_n, $n=1,2,\dots$. Let us introduce the Newtonian potential $\phi\equiv\frac{1}{2}c^2(g_{00}-1)$ where g_{00} is the

relevant component of the metric tensor and c is the velocity of light. We consider the *field* $\phi(r)$ canonical coordinates of the gravitational dynamics and we denote its canonical conjugate momenta by the field $\pi(r)$. The total Hamiltonian of the interacting system takes the following form:

$$H(q,p,\phi,\pi) = H_m(q,p) + \frac{1}{8\pi G}\int_r \left(\frac{1}{c^2}\pi^2 + |\nabla\phi|^2\right) + \int_r f(r)\phi(r) \qquad (20)$$

where G is the Newton constant and $f(r)$ stands for the mass distribution *operator* of the quantized matter [7]. By comparing the third (interaction) term on the RHS of Eq. (20) to the RHS of Eq. (6) one identifies the quantized "current" J_1^α by $f(r)$ and the classical "current" J_2^α by $\phi(r)$ while, obviously, summations over label α will be replaced by integrations over the spatial coordinate r.

Let us substitute the Hamiltonian (20) into the semiclassical master equation (17):

$$\begin{aligned}\dot\rho = -\frac{i}{\hbar}[H_m,\rho] - \frac{i}{\hbar}\int_r \phi(r)[f(r),\rho] \\ -\frac{1}{4\pi G}\int_r\Big(\frac{1}{c^2}\pi(r)\frac{\delta\rho}{\delta\phi(r)} + \Delta\phi(r)\frac{\delta\rho}{\delta\pi(r)}\Big) + \frac{1}{2}\int_r\Big[f(r),\frac{\delta\rho}{\delta\pi(r)}\Big]_+ \\ -\frac{1}{4}\int_r\int_{r'}\lambda(r,r')[f(r),[f(r'),\rho]\,] + \frac{1}{4}\int_r\int_{r'}\lambda^{-1}(r,r')\frac{\delta^2\rho}{\delta\pi(r)\delta\pi(r')}.\end{aligned} \qquad (21)$$

Remember that, in fact, ρ stands for $\rho(\phi,\pi)$ which is a hybrid of density operator for matter and of phase space distribution for gravity. For instance, its functional integral yields the reduced density operator ρ_m of the quantized matter: $\rho_m = \int\int \rho(\phi,\pi)\mathcal{D}\phi\mathcal{D}\pi$. We concentrate on the reduced dynamics of ρ_m. In addition to assuming weak gravity, Newtonian approximation will be considered so that we neglect the term with factor $1/c^2$ on the RHS of Eq. (21) and suppose that matter's quantum state ρ_m determines Newton potential ϕ via the following ansatz:

$$\int \rho(\phi,\pi)\mathcal{D}\pi = \prod_r \delta\Big(\phi(r) + \int_{r'}\frac{G/2}{|r-r'|}[f_+(r') + f_-(r')]\Big)\rho_m. \qquad (22)$$

The subscripts + and $-$ indicate that the operator f is to multiply ρ_m from the left or from the right, respectively. [In such a way hermiticity of ρ is retained by Eq. (22).] Let us integrate both sides of Eq. (21) over the fields ϕ,π while we substitute the ansatz (22) into it. We obtain the following

result:

$$\dot{\rho}_m = -\frac{i}{\hbar}[H_m + H_g, \rho_m] - \frac{1}{4}\int_r\int_{r'} \lambda(r,r')[f(r),[f(r'),\rho_m] \tag{23}$$

where H_g is the well-known Newtonian potential energy:

$$H_g = -\frac{G}{2}\int_r\int_{r'} \frac{f(r)f(r')}{|r-r'|}. \tag{24}$$

One determines the correlation function λ by intuitive considerations. Invoke the interpretation (19). It follows that λ is related to the hypothetical fluctuations of f and ϕ:

$$\begin{aligned} \Big\langle \delta f(r',t')\delta f(r,t)\Big\rangle_{noise} &= \frac{1}{2}\lambda^{-1}(r',r)\delta(t'-t), \\ \Big\langle \delta\phi(r',t')\delta\phi(r,t)\Big\rangle_{noise} &= \frac{\hbar^2}{2}\lambda(r',r)\delta(t'-t). \end{aligned} \tag{25}$$

In Newtonian approximation, the expectation values of the Newton potential ϕ and the mass distribution f are related by the Poisson equation $\Delta\langle\phi(r)\rangle = 4\pi G\langle f(r)\rangle$ as can be seen easily from Eq. (22). *If* the fluctuating "currents", too, satisfied the Poisson equation then Eqs. (25) would lead to the constraint $\Delta\Delta'\lambda(r,r') = (4\pi G)^2\lambda^{-1}(r,r')$. The unique translation invariant correlation satisfying this constraint is $\lambda(r,r') = (G/\hbar)|r-r'|^{-1}$ [8]. So the reduced master equation (21) of the quantized matter takes the form:

$$\dot{\rho}_m = -\frac{i}{\hbar}[H_m + H_g, \rho_m] - \frac{1}{4}\int_r\int_{r'} \frac{G/\hbar}{|r-r'|}[f(r),[f(r'),\rho_m]. \tag{26}$$

This equation represents a simplistic model of semiclassical gravity. The equation itself was first obtained as a result of heuristic efforts to originate macroscopic decoherence from gravitational fluctuations [9]. (Its measurement theoretical aspects are discussed, e.g., in Refs. [10].)

To conclude our Letter we mention less fundamental applications of the proposed semiclassical equation (17), namely for interacting quantum systems whose particular subsystems behave classically at certain special conditions. In each case, the choice of noises is not unique. The issue can be fixed by invoking dimensional, symmetry, or other intuitive considerations (like in the example above). On the other hand, we guess that even a larger class of (non-white) noises should be taken into consideration for instance

in the relativistic regime which is completly beyond the scope of our Letter.

This work was supported by the grants OTKA No. 1822/91 and T016047.

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